

DEGENERATIONS OF RICCI-FLAT CALABI-YAU MANIFOLDS

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ABSTRACT. This paper is a sequel to [16]. We further investigate the Gromov-Hausdorff convergence of Ricci-flat Kähler metrics under degenerations of Calabi-Yau manifolds. We extend Theorem 1.1 in [16] by removing the condition on existence of crepant resolutions for Calabi-Yau varieties.

1. INTRODUCTION

A Calabi-Yau manifold M is a projective manifold with trivial canonical bundle $\mathcal{K}_M \cong \mathcal{O}_M$. Yau's theorem for Calabi conjecture (cf. [26]) asserts that, for any Kähler class $\alpha \in H^{1,1}(M, \mathbb{R})$, there exists a unique Ricci-flat Kähler metric g on M whose Kähler form ω represents α . The metric behavior of Ricci-flat Calabi-Yau manifolds is studied by various authors from many perspectives (cf. [1], [4], [9], [10], [13], [14], [16], [17], [20], [22], [23], [19], [25]). The present paper is a sequel to [16], and we further investigate the Gromov-Hausdorff convergence of Ricci-flat Kähler metrics under degenerations of Calabi-Yau manifolds.

The moduli space \mathfrak{M} of a polarized Calabi-Yau manifold (M, L) , i.e. a Calabi-Yau manifold M with an ample line bundle L , exists, and is a quasi-projective variety (cf. [24]). Yau's theorem can be viewed as a map

$$\mathcal{CY} : \mathfrak{M} \longrightarrow \mathcal{X} \quad \text{by } (M, L) \mapsto (M, g_L),$$

where g_L is a Ricci-flat Kähler metric on M whose Kähler class represents $c_1(L)$, and \mathcal{X} denotes the space of isometric classes of compact metric spaces with Gromov-Hausdorff topology. An interesting question is to understand the relationship between the closure of $\mathcal{CY}(\mathfrak{M})$ in \mathcal{X} and the natural algebro-geometric compactification of \mathfrak{M} .

Assume that \mathcal{M} is an $(n+1)$ -dimensional variety, $\pi : \mathcal{M} \rightarrow \Delta$ is a proper flat morphism from \mathcal{M} to a disc $\Delta \subset \mathbb{C}$, $M_0 = \pi^{-1}(0)$ is a singular projective

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variety, $M_t = \pi^{-1}(t)$ is a smooth projective n -dimensional manifold for any $t \in \Delta \setminus \{0\}$, and the relative canonical bundle $\mathcal{K}_{\mathcal{M}/\Delta}$ is trivial on $\mathcal{M} \setminus M_0$. Then for any $t \in \Delta \setminus \{0\}$ $M_t = \pi^{-1}(t)$ is a Calabi-Yau n -manifold, and (\mathcal{M}, π) is called a *degeneration* of Calabi-Yau manifolds over the disc Δ . If \mathcal{L} is an ample line bundle on \mathcal{M} whose differential type is independent of t , then there is a map from $\Delta \setminus \{0\}$ to the moduli space \mathfrak{M} of $(M_t, \mathcal{L}|_{M_t})$, and $0 \in \Delta$ corresponds to a point in the algebro-geometric compactification of \mathfrak{M} . A Calabi-Yau variety is a projective Gorenstein normal variety M_0 with trivial canonical sheaf $\mathcal{K}_{M_0} \cong \mathcal{O}_{M_0}$ and only canonical singularities.

In this paper we study the Gromov-Hausdorff limit of $\mathcal{CY}((M_t, \mathcal{L}|_{M_t}))$ when $0 \in \Delta$ represents a Calabi-Yau variety. The main result is:

Theorem 1.1. *Let (\mathcal{M}, π) be a degeneration of Calabi-Yau manifolds over a disc $\Delta \subset \mathbb{C}$, and let \mathcal{L} be an ample line bundle on \mathcal{M} . Assume that M_0 is a Calabi-Yau n -variety ($n \geq 2$) with singular set S , and the relative canonical bundle $\mathcal{K}_{\mathcal{M}/\Delta}$ of (\mathcal{M}, π) is trivial, i.e., $\mathcal{K}_{\mathcal{M}/\Delta} \cong \mathcal{O}_{\mathcal{M}}$. If \tilde{g}_t denotes the unique Ricci-flat Kähler metric with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$, $t \in \Delta \setminus \{0\}$, then*

$$(M_t, \tilde{g}_t) \xrightarrow{d_{GH}} (X, d_X),$$

when $t \rightarrow 0$, in the Gromov-Hausdorff sense, where (X, d_X) denotes the metric completion of $(M_0 \setminus S, d_g)$, g is a Ricci-flat Kähler metric on $M_0 \setminus S$, and d_g is the Riemannian distance function of g . Furthermore, $X \setminus (M_0 \setminus S)$ is a closed subset of Hausdorff dimension less or equal to $2n - 4$, and any tangent cone $T_x X$, $x \in X \setminus (M_0 \setminus S)$, is not \mathbb{R}^{2n} .

In [16], Theorem 1.1 is proved under an additional assumption that M_0 admits a crepant resolution, i.e. a resolution $\bar{\pi} : \bar{M} \rightarrow M_0$ with $\bar{\pi}^* \mathcal{K}_{M_0} = \mathcal{K}_{\bar{M}}$. However, many Calabi-Yau varieties do not admit any crepant resolution, e.g. Calabi-Yau varieties of dimension 4 with only finite ordinary double points as singularities.

In [16], it is proved under the hypothesis of Theorem 1.1 that there is a uniform upper bound $D > 0$ for the diameter of (M_t, \tilde{g}_t) , i.e.

$$\text{diam}_{\tilde{g}_t}(M_t) \leq D,$$

and furthermore, $F_t^* \tilde{g}_t$ converges to a Ricci-flat Kähler metric g in the local C^∞ -sense on $M_0 \setminus S$, when $t \rightarrow 0$, where $F_t : M_0 \setminus S \rightarrow M_t$ is a smooth family of embeddings with $F_0 = \text{id}$. However, very little is known about the metric behavior near singularities, and the global convergence. The Gromov precompactness theorem (cf. [8]) asserts that, for any sequence $t_k \rightarrow 0$, there is a subsequence $(M_{t_k}, \tilde{g}_{t_k})$ converging to a compact length metric space (X, d_X) in the Gromov-Hausdorff sense. Moreover, the structure of the limit metric space X is studied by Cheeger Colding and Tian (cf. [2] [3] [4]), and it is shown that there is a closed subset $S_X \subset X$ of Hausdorff dimension less or equal to $2n - 4$ such that any tangent cone $T_x X$, $x \in S_X$, is not \mathbb{R}^{2n} .

and $X \setminus S_X$ is a smooth manifold admitting a Ricci-flat Kähler metric g_∞ . These results will play a role in the proof of Theorem 1.1.

In Theorem 1.1, we obtain that (X, d_X) is isometric to the metric completion of $(M_0 \setminus S, d_g)$, and the Gromov-Hausdorff convergence takes place for the whole continuous parameter $t \in \Delta \setminus \{0\}$, i.e. the Gromov-Hausdorff limit X is unique and thus is independent of the choice of subsequences of t . Furthermore, $(X \setminus S_X, g_\infty)$ is isometric to $(M_0 \setminus S, g)$.

Finally, we remark that after this paper is completed, we notice that Theorem 1.2 in a preprint [5] posted a few days ago (see also [21]) implies that X is homeomorphic to M_0 . We also mention that the homeomorphism property was proved earlier for K3 surfaces in [13], and for Calabi-Yau threefolds with only finite ordinary double points as singularities in [19] by assuming the existence of crepant resolutions.

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2. PROOF OF THEOREM 1.1

Let (\mathcal{M}, π) be a degeneration of Calabi-Yau manifolds with trivial relative canonical bundle $\mathcal{K}_{\mathcal{M}/\Delta}$, and let \mathcal{L} be an ample line bundle on \mathcal{M} . Then there is an embedding $\mathcal{M} \hookrightarrow \mathbb{CP}^N \times \Delta$ such that $\mathcal{L}^m = \mathcal{O}_\Delta(1)|_{\mathcal{M}}$ for some $m \geq 1$, π is a proper surjection given by the restriction of the projection from $\mathbb{CP}^N \times \Delta$ to Δ , and the rank of π_* is 1 on $\mathcal{M} \setminus S$. Denote

$$\omega_t = \frac{1}{m} \omega_{FS}|_{M_t},$$

where ω_{FS} is the standard Fubini-Study metric on \mathbb{CP}^N , and g_t is the corresponding Kähler metric of ω_t . Let Ω_t be a relative holomorphic volume form, i.e., a nowhere vanishing section of $\mathcal{K}_{\mathcal{M}/\Delta}$. Yau's theorem of Calabi's conjecture ([26]) asserts that there is a unique Ricci-flat Kähler metric \tilde{g}_t with Kähler form $\tilde{\omega}_t \in [\omega_t] = c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$ for $t \in \Delta \setminus \{0\}$, i.e., there is a unique function φ_t on M_t satisfying that $\tilde{\omega}_t = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_t$, and

$$(2.1) \quad (\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n = (-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \bar{\Omega}_t, \quad \text{with} \quad \sup_{M_t} \varphi_t = 0$$

where σ_t is a constant depending only on t . By Lemma 3.1 in [16],

$$(2.2) \quad \|\varphi_t\|_{C^0} \leq \bar{C},$$

for a constant \bar{C} independent of t , and by Lemma 3.2 in [16],

$$(2.3) \quad \omega_t \leq C \tilde{\omega}_t,$$

for a constant $C > 0$ independent of t .

Denote $f_t : (M_t, \tilde{\omega}_t) \longrightarrow (M_t, \omega_t) \subset (\mathbb{CP}^N, \frac{1}{m}\omega_{FS})$ the inclusion map induced by $\mathcal{M} \subset \mathbb{CP}^N \times \Delta$. Note that f_t is holomorphic, i.e. $\bar{\partial}f_t = 0$, $f_t^* \frac{1}{m}\omega_{FS} = \omega_t$ and

$$(2.4) \quad |df_t|_{\tilde{g}_t, g_{FS}}^2 = 2|\partial f_t|_{\tilde{\omega}_t, \omega_{FS}}^2 = 2\text{tr}_{\tilde{\omega}_t} f_t^* \omega_{FS} = 2m\text{tr}_{\tilde{\omega}_t} \omega_t \leq C$$

on M_t . Here the last inequality of (2.4) follows from (2.3). So f_t is a family of Lipschitz maps with a uniform Lipschitz constant independent of t .

By Theorem 1.4 in [16], the diameter of (M_t, \tilde{g}_t) is uniformly bounded, i.e.

$$\text{diam}_{\tilde{g}_t}(M_t) \leq D,$$

for a constant D independent of $t \in \Delta \setminus \{0\}$. Furthermore, for any smooth family of embeddings $F_t : M_0 \setminus S \rightarrow M_t$ with $F_0 = \text{id}$,

$$F_t^* \tilde{g}_t \longrightarrow g, \quad F_t^* \tilde{\omega}_t \longrightarrow \omega, \quad \varphi_t \circ F_t \longrightarrow \varphi_0,$$

when $t \rightarrow 0$, in the C^∞ -sense on any compact subset $K \subset M_0 \setminus S$, where φ_0 is a smooth function, ω is a Ricci-flat Kähler form on $M_0 \setminus S$ obtained in [6], and g is the corresponding Kähler metric of ω , i.e. $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0$ and

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n = (-1)^{\frac{n^2}{2}} e^{\sigma_0} \Omega_0 \wedge \bar{\Omega}_0,$$

on $M_0 \setminus S$. By [30], φ_0 extends to a continuous function on M_0 , still denoted by φ_0 .

Proof of Theorem 1.1. By Gromov compactness theorem (cf. [8]), there is a subsequence $(M_{t_k}, \tilde{g}_{t_k})$ converging to a compact length metric space (X, d_X) in the Gromov-Hausdorff sense, i.e. there exists a $\epsilon(k)$ -approximation $\phi_k : M_{t_k} \rightarrow X$ where $\epsilon(k) \rightarrow 0$ when $k \rightarrow \infty$.

Recall that the convergence for Riemannian manifolds with Ricci-curvature bounded from below was intensively studied by Cheeger and Colding (cf. [2] [3] etc.). Applying Section 6 and 7 of [2] to our circumstances, we conclude that there is a closed subset $S_X \subset X$ of Hausdorff dimension $\dim_{\mathcal{H}} S_X \leq 2n - 2$ such that, for any $x \in S_X$, the tangent cone $T_x X$ is not isometric to \mathbb{R}^{2n} . Furthermore, $X \setminus S_X$ is a smooth open complex manifold, and $d_X|_{X \setminus S_X}$ is induced by a Ricci-flat Kähler metric g_∞ on $X \setminus S_X$. By Section 7 of [2], \tilde{g}_{t_k} smoothly converges to g_∞ on $X \setminus S_X$ under the Gromov-Hausdorff convergence of $(M_{t_k}, \tilde{g}_{t_k})$ to (X, d_X) , i.e., there are compact subsets $K_1 \subset \cdots \subset K_k \subset K_{k+1} \subset \cdots \subset X \setminus S_X$ such that $\bigcup_{k=1}^\infty K_k = X \setminus S_X$, $\phi_k|_{\phi_k^{-1}(K_k)} : \phi_k^{-1}(K_k) \rightarrow K_k$ can be chosen as diffeomorphisms and

$$\phi_k^{-1}|_{K_k}^* \tilde{g}_{t_k} \longrightarrow g_\infty, \quad \phi_{k,*} J_{t_k} \phi_k^{-1} \longrightarrow J_\infty$$

on any compact subset $K \subset X \setminus S_X$ in the C^∞ -sense, where J_{t_k} (resp. J_∞) denotes the complex structure of M_{t_k} (resp. $X \setminus S_X$).

The same argument as in the proof of Lemma 4.1 in [17] implies that, by passing to a subsequence, $\phi_k \circ F_{t_k} : M_0 \setminus S \rightarrow X$ converges to a local isometric embedding $\Psi : (M_0 \setminus S, g) \rightarrow (X, d_X)$, i.e. for a $y \in M_0 \setminus S$, there is a $\rho_y > 0$ such that $d_X(\Psi(y_1), \Psi(y_2)) = d_g(y_1, y_2)$ for any $y_1, y_2 \in B_g(y, \rho_y)$.

Moreover, for any $y \in M_0 \setminus S$, $F_{t_k}(y) \in M_{t_k}$ converges to a point $x \in X$ under the Gromov-Hausdorff convergence of $(M_{t_k}, \tilde{g}_{t_k})$ to (X, d_X) , and $\Psi(y) = x$.

Lemma 2.1. *A subsequence of $f_{t_k} : M_{t_k} \rightarrow \mathbb{CP}^N$ converges to a continuous map $f_\infty : X \rightarrow \mathbb{CP}^N$ which satisfies that $f_\infty(X) = M_0$, $f_\infty \circ \Psi = \text{id} : M_0 \setminus S \rightarrow M_0 \setminus S$, and $f_\infty|_{X \setminus S_X}$ is holomorphic. Furthermore,*

$$f_{t_k} \circ \phi_k^{-1}|_K \longrightarrow f_\infty|_K,$$

in the C^∞ -sense on any compact subset $K \subset X \setminus S_X$.

Proof. By (2.4), f_{t_k} are Lipschitz maps with a uniform Lipschitz constant independent of k . Hence, passing to a subsequence, $f_{t_k} : M_{t_k} \rightarrow \mathbb{CP}^N$ converges to a continuous map $f_\infty : X \rightarrow \mathbb{CP}^N$ when $t_k \rightarrow 0$ (cf. [15]), i.e. for any sequence $p_k \in M_{t_k}$ which converges to $x \in X$ under the Gromov-Hausdorff convergence of $(M_{t_k}, \tilde{g}_{t_k})$ to (X, d_X) , we have that $f_{t_k}(p_k)$ converges to $f_\infty(x)$ in \mathbb{CP}^N . From that $f_{t_k}(M_{t_k}) = M_{t_k} \subset \mathbb{CP}^N$, we see that $f_\infty(X) = M_0 \subset \mathbb{CP}^N$. Note that for any $y \in M_0 \setminus S$, $F_{t_k}(y) \in M_{t_k}$ converges to a point $x \in X$ under the Gromov-Hausdorff convergence of $(M_{t_k}, \tilde{g}_{t_k})$ to (X, d_X) , and $F_{t_k}(y)$ converges to y in \mathbb{CP}^N . By the convergence of f_{t_k} , $f_{t_k}(F_{t_k}(y)) = F_{t_k}(y)$ and $F_0 = \text{id}$, we have that $f_\infty(x) = y$. Thus $f_\infty \circ \Psi = \text{id} : M_0 \setminus S \rightarrow M_0 \setminus S$.

We claim that $f_\infty|_{X \setminus S_X}$ is holomorphic. Note that $f_{t_k} : M_{t_k} \rightarrow \mathbb{CP}^N$ is holomorphic, i.e.

$$\bar{\partial} f_{t_k} = \frac{1}{2}(df_{t_k} + J \circ df_{t_k} \circ J_{t_k}) = 0,$$

where J denotes the complex structure of \mathbb{CP}^N . Then $f_{t_k} : (M_{t_k}, \tilde{g}_{t_k}) \rightarrow (\mathbb{CP}^N, g_{FS})$ is a harmonic map, and by (2.4)

$$|df_{t_k}|_{\tilde{g}_{t_k}, g_{FS}}^2 \leq C$$

for a constant $C > 0$ independent of k . In local coordinates on M_{t_k} and \mathbb{CP}^N we have

$$\Delta_{\tilde{g}_{t_k}} f_{t_k}^i + \tilde{g}_{t_k}^{\alpha\beta} \Gamma_{jl}^i \frac{\partial f_{t_k}^j}{\partial x^\alpha} \frac{\partial f_{t_k}^l}{\partial x^\beta} = 0,$$

where Γ_{jl}^i denotes the Christoffel symbols of g_{FS} (cf. [18]). On any compact subset $K \subset X \setminus S_X$, $\|f_{t_k}^i\|_{C^{1,\alpha}} \leq C_K$ for a constant $C_K > 0$ independent of k by the standard Schauder estimates (cf [11]), since $\phi_k^{-1}|_{K_k}^* \tilde{g}_{t_k}$ converges to g_∞ in the C^∞ -sense, and $|\frac{\partial f_{t_k}^j}{\partial x^\alpha}|_{C^0} \leq C$. Here we identify K and $\phi_k^{-1}(K)$ for $k \gg 1$. The elliptic bootstrap estimates show that $\|f_{t_k}^i\|_{C^l} \leq C_{l,K}$ for any $l \in \mathbb{N}$ on K . Hence, passing a subsequence, f_{t_k} converges to f_∞ smoothly on K , and, by the convergence of J_{t_k} to J_∞ , f_∞ is holomorphic, i.e.

$$\bar{\partial} f_\infty = \frac{1}{2}(df_\infty + J \circ df_\infty \circ J_\infty) = 0.$$

By the standard diagonal argument, f_∞ is holomorphic on $X \setminus S_X$. □

Lemma 2.2. $f_\infty(X \setminus S_X) = M_0 \setminus S$.

Proof. For any $x \in \Psi(M_0 \setminus S)$, the tangent cone $T_x X \cong \mathbb{R}^{2n}$ since Ψ is a local isometry. Thus $\Psi(M_0 \setminus S) \subset X \setminus S_X$. If we denote $E = X \setminus (S_X \cup \Psi(M_0 \setminus S))$, then $f_\infty(E) \subset S$ by $f_\infty \circ \Psi = \text{id}$ and $f_\infty(X) = M_0$. It suffices to show that E is empty.

Assume that E is not empty. Since $f_\infty|_{X \setminus S_X}$ is holomorphic, $E = f_\infty|_{X \setminus S_X}^{-1}(S)$ is a complex subvariety of $(X \setminus S_X, J_\infty)$, and $f_\infty|_{X \setminus (S_X \cup E)} : X \setminus (S_X \cup E) \rightarrow M_0 \setminus S$ is bi-holomorphic. For any $y \in S \cap f_\infty(X \setminus S_X) = f_\infty(E)$, $E_y = f_\infty^{-1}(y) \cap X \setminus S_X$. Since M_0 is normal, by Zariski's main theorem (cf [12]) E_y is a connected subvariety. If $\dim_{\mathbb{C}} E_y = 0$, for all $y \in S \cap f_\infty(X \setminus S_X)$, then E_y is a point, and the Hartogs' theorem shows that Ψ can extend to a holomorphic map from $f_\infty(X \setminus S_X)$ to $X \setminus S_X$. Hence $f_\infty|_{X \setminus S_X} : X \setminus S_X \rightarrow f_\infty(X \setminus S_X)$ is bi-holomorphic, which contradicts to $y \in S$ is a singular point. Now we can assume that $\dim_{\mathbb{C}} E_y \geq 1$ for a $y \in S \cap f_\infty(X \setminus S_X)$.

Since f_∞ is holomorphic on $X \setminus S_X$, $f_\infty^* \omega_0 = \frac{1}{m} f_\infty^* \omega_{FS}$ is a smooth $(1, 1)$ -form on $X \setminus S_X$. Since Ψ is local isometric embedding and $f_\infty \circ \Psi = \text{id}$, $f_\infty = \Psi^{-1}$ and $f_\infty^* \omega = \omega_\infty$ on $X \setminus (S_X \cup E)$ respectively. By Lemma 2.1, $\phi_k^{-1,*} f_{t_k}^* \omega_{t_k} = \frac{1}{m} \phi_k^{-1,*} f_{t_k}^* \omega_{FS}$ converges smoothly to $f_\infty^* \omega_0 = \frac{1}{m} f_\infty^* \omega_{FS}$ on any compact subset $K \subset X \setminus S_X$. Note that $\tilde{\omega}_{t_k} = \omega_{t_k} + \sqrt{-1} \partial \bar{\partial} \varphi_{t_k}$ with $\|\varphi_{t_k}\|_{C^0} \leq \bar{C}$ for a constant \bar{C} independent of k (See (2.2)). Then

$$\Delta_{\tilde{\omega}_{t_k}}(\varphi_{t_k} \circ f_{t_k}) = n - \text{tr}_{\tilde{\omega}_{t_k}} f_{t_k}^* \omega_{t_k}.$$

As seen in the proof of Lemma 2.1, $\phi_k^{-1,*} \tilde{\omega}_{t_k}$ smoothly converges to ω_∞ on any compact subset $K \subset X \setminus S_X$. By standard elliptic estimates, we have

$$\|\varphi_{t_k} \circ f_{t_k} \circ \phi_k^{-1}\|_{C^l(K)} \leq C_{l,K},$$

for any $l \in \mathbb{N}$ and a constant $C_{l,K} > 0$ independent of k . Hence there is a smooth function φ_∞ on $X \setminus S_X$ such that, passing to a subsequence $\varphi_{t_k} \circ f_{t_k} \circ \phi_k^{-1}$ converges smoothly to φ_∞ on K , and

$$\omega_\infty = f_\infty^* \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\infty.$$

By Lemma 2.1, for any $x \in X \setminus (S_X \cup E)$, $f_{t_k} \circ \phi_k^{-1}(x)$ converges to $f_\infty(x)$ in \mathbb{CP}^N , and $f_\infty(x) \in M_0 \setminus S$. Since $F_{t_k}(f_\infty(x))$ converges to $f_\infty(x)$ in \mathbb{CP}^N , and $\varphi_{t_k} \circ F_{t_k}$ converges to φ_0 on $M_0 \setminus S$ in the C^∞ -sense, we have

$$\varphi_\infty(x) = \lim_{k \rightarrow \infty} \varphi_{t_k}(f_{t_k} \circ \phi_k^{-1}(x)) = \lim_{k \rightarrow \infty} \varphi_{t_k}(F_{t_k}(f_\infty(x))) = \varphi_0(f_\infty(x)).$$

Thus $\varphi_\infty = \varphi_0 \circ f_\infty$ on $X \setminus (S_X \cup E)$. Note that φ_0 is a continuous function on M_0 , and, hence, $\varphi_0 \circ f_\infty$ is a continuous function on $X \setminus S_X$. We obtain that $\varphi_\infty = \varphi_0 \circ f_\infty$ on $X \setminus S_X$, and $\varphi_\infty|_{E_y} = \varphi_0(y)$ is a constant function, which implies $\omega_\infty|_{E_y} \equiv 0$. It is a contradiction, and thus E is empty. \square

Finally, we finish the proof. By Section 3 of [3], we obtain that for any $x_1, x_2 \in \Psi(M_0 \setminus S) = X \setminus S_X$ and any $\delta > 0$, there is a curve γ_δ connecting

x_1 and x_2 in $\Psi(M_0 \setminus S)$ such that

$$d_X(x_1, x_2) \leq \text{length}_{d_X}(\gamma_\delta) \leq \delta + d_X(x_1, x_2).$$

Thus (X, d_X) is isometric to the metric completion of $(M_0 \setminus S, d_g)$. By [4], the Hausdorff dimension $\dim_{\mathcal{H}} S_X \leq 2n - 4$. \square

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